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LETTER TO THE EDITOR

Topological horseshoe and numerically observed chaotic behaviour in the Henon mapping

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Abstract. We prove the existence of a topological horseshoe in the Henon mapping for values of the parameters such that a strange attractor is numerically observed.

Since Henon (1976) many authors have recently studied the dynamics of the diffeomorphisms of the plane

$$F_{a,b}: (x, y) \to (1 - ax^2 + y, bx). \tag{1}$$

Interest in these mappings has been stimulated by Henon's numerical evidence for a 'strange attractor' when $a = 1{\cdot}4$ and $b = 0{\cdot}3$. Such a behaviour has also been observed for different values of a and b (Feit 1978, Simó 1978, Curry 1979). Nevertheless the actual existence of a strange attractor is far from being proved: it has been proposed (Newhouse 1977) that what Henon did observe is merely a long periodic orbit.

A simpler question is: does there exist chaotic behaviour—e.g. transverse homoclinic points or horseshoe effects: Smale (1967), Nitecki (1971)—in (1)?

Numerical evidence has been given for the existence of transverse homoclinic points by Curry (1979) for Henon's values of a and b, and by Simó (1978) for lower values of a. Analytical proofs have been obtained by Marotto (1979) for b small enough, and by Devaney and Nitecki (1979) for any b and a larger than $2(1 + |b|)^2$. However, neither of these works allows us to ensure the existence of homoclinic points for values of a and b such that a strange attractor is observed for (1): Marotto does not provide an explicit range of b values for which his theorem applies, and the values of a considered by Devaney and Nitecki are such that almost every point in $\mathbb{R}^2$ has an orbit which diverges.

In the present Letter we give a method to prove that a topological horseshoe (Devaney and Nitecki 1979) is imbedded in the dynamics of (1) (this implies the existence of homoclinic points) for parameter values such that a chaotic behaviour is numerically displayed.

When the non-wandering set of $F_{a,b}$ is non-void ($(1 - b)^2 + 4a \geq 0$), this diffeomorphism is topologically conjugate to

$$G_{R,b}: (x, y) \to (y, Ry(1 - y) + bx), \tag{2}$$

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a conjugacy homeomorphism being

$$(x, y) \to (\alpha y + \beta, \alpha' x + \beta'), \tag{3}$$

where

$$R = 1 - b + [(1-b)^2 + 4a]^{1/2} \tag{4}$$

and

$$\alpha = R/a, \qquad \alpha' = b\alpha, \qquad \beta = -R/2a, \qquad \beta' = b\beta. \tag{5}$$

When $b = 0$, $G_{R,0}|_Q$ is an endomorphism of the square $Q = [0, 1] \times [0, 1]$ which maps Q on the graph of $f_R(x) = Rx(1-x)$ restricted to $[0, 1]$. Before considering more general situations, let us restrict ourselves to the case

$$0 < R < 4, \qquad 0 < b < 1 - R/4. \tag{6}$$

Then

$$G_{R,b}(Q) \subset Q \tag{7}$$

and we can obtain a good geometric comprehension of the global structure of the 'strange attractors' of (2) in the following way:

(i) Let us denote

$$\begin{aligned} I_1 &= \{(x, y): x = 0, 0 \leq y \leq 1\}, \\ I_2 &= \{(x, y): x = 0, 0 \leq y \leq b\}, \\ I_3 &= \{(x, y): x = 1, 0 \leq y \leq 1\}, \\ I_4 &= \{(x, y): x = 1, 0 \leq y \leq b\}. \end{aligned} \tag{8}$$

Then the frontier of $G_{R,b}(Q)$ is $G_{R,b}(I_1) \cup I_2 \cup G_{R,b}(I_3) \cup I_4$ (see figure 1(*a*)).

(ii) In order to have a better understanding of the structure of $G^2_{R,b}(Q)$, it is quite natural to use the change of coordinates

$$(X, Y) = (x, [y - Rx(1-x)]/b), \tag{9}$$

which brings $G_{R,b}(Q)$ on the square Q. In these new coordinates $G_{R,b}$ reads

$$(X, Y) \to (RX(1-X) + bY, X), \tag{10}$$

which is precisely (2) with the roles of X and Y exchanged (figure 1(*b*)).

(iii) By iterating this procedure we can draw the first approximations of $\bigcap_{n=0} G^n_{R,b}(Q)$ which contain the attractors (they may be strange) of $G_{R,b}$ (figure 1(*c*), (*d*)).

We now come to the problem of the existence of a horseshoe for (2). We first state a theorem when condition (6) is satisfied.

Theorem. There is a value $R_0 \sim 3{\cdot}6785735$† defined by $f^3_{R_0}(0{\cdot}5) = (R_0 - 1)/R_0$ such that, for $R_0 < R < 4$ and $0 < b < 1 - R/4$, there is a total topological horseshoe imbedded in the dynamics.

† R_0 is the value studied by Ruelle (1977). It may be interpreted as the value of R corresponding to a first tangent homoclinic point (in the sense of Block (1978)) occurring in the unstable manifold of the non-trivial fixed point $x^* = (R-1)/R$ of $f_R(x) = Rx(1-x)$.

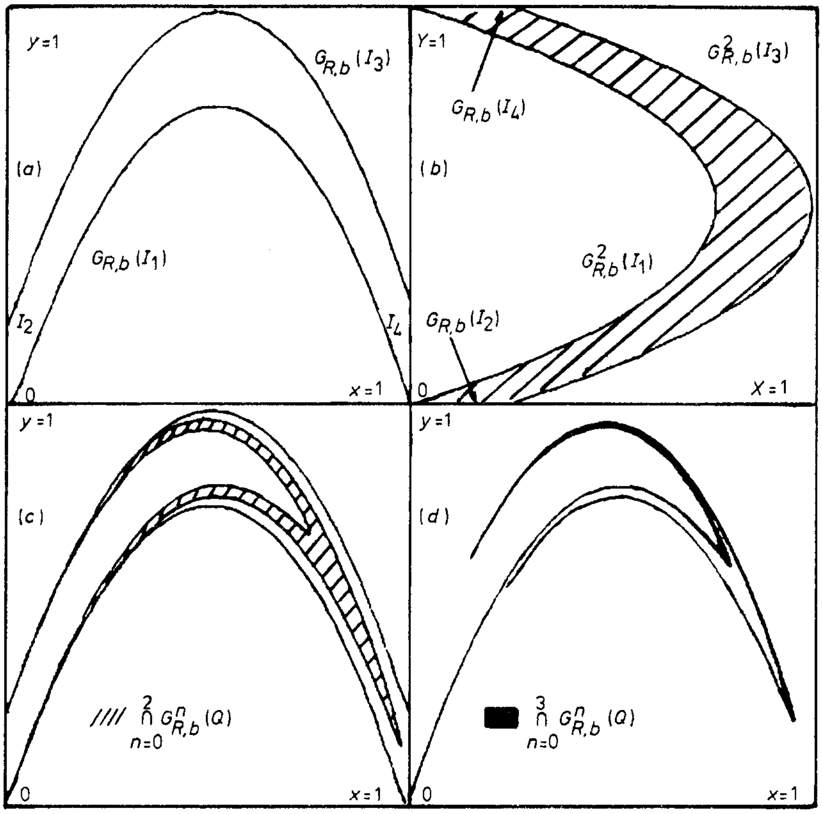

Figure 1. The first steps in the geometrical construction of $\bigcap_{n=0} G^n_{R,b}(Q)$.

Once a horseshoe is exhibited, the further analysis up to the existence of homoclinic points is classical, and we refer to Smale (1967), Nitecki (1971) and Devaney and Nitecki (1979) for a comprehensive treatment. The horseshoe effect is represented in figure 2 for $R = 3{\cdot}7$ and $b = 0{\cdot}07$. In the same figure we also show the 'strange attractor' for the same values of the parameters.

Using the above remarks on the geometry of the successive iterates of Q under $G_{R,b}$, we now outline the construction of a 'support' ABCD for the topological horseshoe; ABCD is defined as follows (figure 2):

$$
\begin{aligned}
&\text{AB is a horizontal line } \{(x, y) | x_A \leq x \leq x_B,\ y = y_A\},\\
&\text{BC is an arc of parabola } \{(x, y) | x_C \leq x \leq x_B,\ y = Rx(1-x) + b\},\\
&\text{CD is a horizontal line } \{(x, y) | x_D \leq x \leq x_C,\ y = y_C\},\\
&\text{DA is an arc of parabola } \{(x, y) | x_D \leq x \leq x_A,\ y = Rx(1-x)\}.
\end{aligned}
\tag{11}
$$

The proof of the above-mentioned theorem consists of showing that there exist A, B, C, D such that $G^2_{R,b}|_{ABCD}$ is actually a topological horseshoe. When (6) is verified, this is achieved in three steps which involve a long but trivial series of majorations.

Step 1. C, D are defined by

$$
y_C = y_D = x_C = Rx_C(1-x_C) + b = Rx_D(1-x_D). \tag{12}
$$

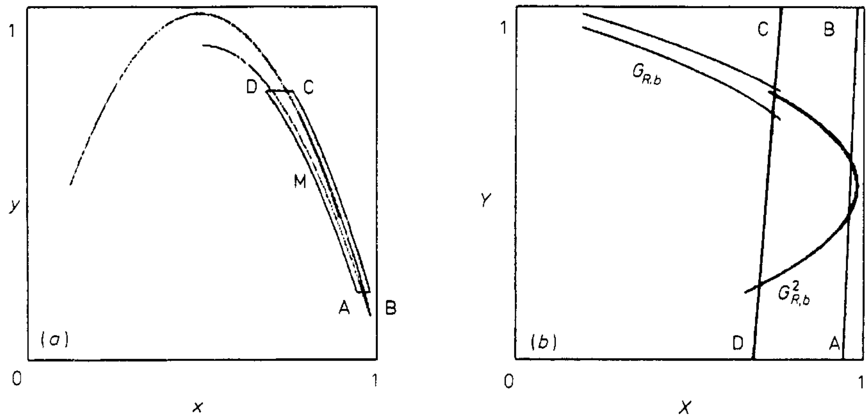

Figure 2. (*a*) In the (x, y) coordinates, the strange attractor for $R = 3{\cdot}7$ and $b = 0{\cdot}07$ is represented, together with the support ABCD of the topological horseshoe; the point M is defined in the main text. (*b*) For the same parameter values, ABCD, $G_{R,b}(\mathrm{ABCD})$ and $G^2_{R,b}(\mathrm{ABCD})$ are represented in the (X, Y) coordinates. $G^2_{R,b}|_{\mathrm{ABCD}}$ is a topological horseshoe.

Then for $\mathrm{P} \in \mathrm{CD}$

$$y_{G^2_{R,b}(\mathrm{P})} > y_{\mathrm{C}} \qquad \text{or} \qquad x_{G^2_{R,b}(\mathrm{P})} < 0{\cdot}5. \tag{13}$$

Step 2. In order to prove that the bend of $G^2_{R,b}(\mathrm{ABCD})$ is out of ABCD as presented in figure 2, we introduce the point M as defined by

$$y_{\mathrm{M}} = 0{\cdot}5 = Rx_{\mathrm{M}}(1 - x_{\mathrm{M}}). \tag{14}$$

Then either

$$y_{G^2_{R,b}(\mathrm{N})} > y_{\mathrm{C}} \qquad \text{or} \qquad x_{G^2_{R,b}(\mathrm{N})} < 0{\cdot}5, \tag{15}$$

where

$$y_{\mathrm{N}} = Rx_{\mathrm{N}}(1 - x_{\mathrm{N}}) + b = y_{G^2_{R,b}(\mathrm{M})}. \tag{16}$$

Step 3. (14), (15) and (16) imply that there exist A and B such that on one hand the bend of $G^2_{R,b}(\mathrm{ABCD})$ is out of ABCD, i.e.

$$y_{\mathrm{B}} = y_{\mathrm{A}} > y_{G^2_{R,b}(\mathrm{M})}, \tag{17}$$

and on the other hand $G^2_{R,b}(\mathrm{ABCD})$ crosses ABCD twice, i.e. for $\mathrm{P} \in \mathrm{AB}$ the condition (13) is satisfied. From (11) we deduce that

$$y_{\mathrm{B}} = y_{\mathrm{A}} = Rx_{\mathrm{A}}(1 - x_{\mathrm{A}}) = Rx_{\mathrm{B}}(1 - x_{\mathrm{B}}) + b. \tag{18}$$

For other values of R and b which are not restricted by condition (6), a specific support (depending on R and b values) forn a horseshoe can still be defined. In figure 3 a polygon ABCD is drawn, such that $G^2_{R,b}|_{\mathrm{ABCD}}$ is a topological horseshoe when $b = 0{\cdot}3$ and $R = 0{\cdot}7 + \sqrt{6{\cdot}09} \simeq 3{\cdot}16779\ldots$, which correspond to Henon (1976) values of the parameters.

To conclude this Letter let us emphasise that the chaotic behaviour numerically observed for a given map F (real endomorphism or diffeomorphism of the plane) always

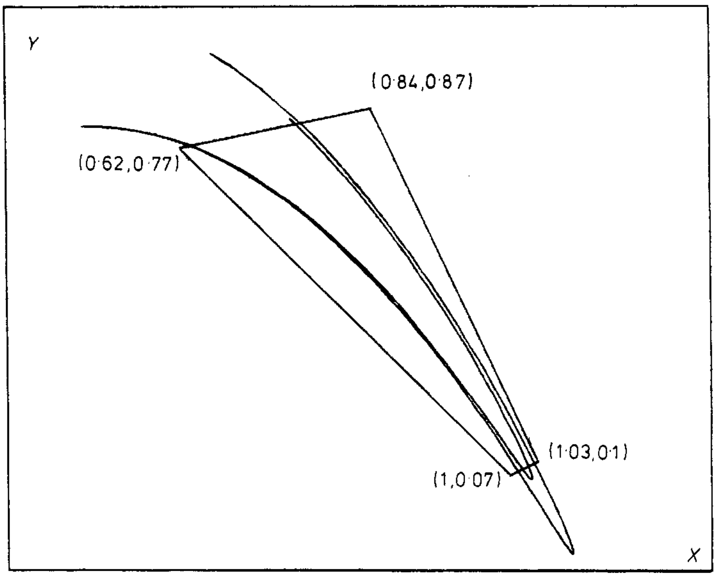

Figure 3. A polygon such that $G^2_{R,b}|_{ABCD}$ is a topological horseshoe for R and b corresponding to Henon's parameter values.

seems to be associated with the existence of a family $\{Q_\alpha\}$ of polygons such that: (1) for each α there exists $n(\alpha)$ such that $F^{n(\alpha)}|_{Q_\alpha}$ is a topological horseshoe; (2) $F^{n(\alpha)}|_{Q_\alpha} - Q_\alpha$ is eventually mapped in the set $\bigcup_\alpha Q_\alpha$.

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